THE NORMAL BUNDLE OF A RATIONAL CURVE ON A GENERIC QUINTIC THREEFOLD

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Abstract. This is another proof of the same result in [9]. Let X_0 be a generic quintic hypersurface in $\mathbf{P}^4$ over $\mathbb{C}$ and c_0 a regular map $\mathbf{P}^1 \to X_0$ that is generically one-to-one to its image. In this paper, we show
(1) c_0 must be an immersion, i.e. the differential $(c_0)_* : T_t\mathbf{P}^1 \to T_{c_0(t)}X_0$ is injective at each $t \in \mathbf{P}^1$,
(2) the normal bundle of c_0 satisfies

$$H^1(N_{c_0/X_0}) = 0.$$

1 Introduction

Throughout the paper, we work over $\mathbb{C}$. Let X_0 be a generic quintic threefold in $\mathbf{P}^4$ over $\mathbb{C}$. Let $c_0 : \mathbf{P}^1 \to X_0$ be a birational map onto its image. The regular map $c_0 : \mathbf{P}^1 \to \mathbf{X}_0$ induces a differential map

$$(c_0)_*|_t : T_t\mathbf{P}^1 \quad \to \quad T_{c_0(t)}X_0 \tag{1.1}$$

point-wisely, which induces an injective morphism on the sheaf module, denoted by $(c_0)_*$

$$(c_0)_* : T_{\mathbf{P}^1} \quad \to \quad c_0^*(T_{X_0}). \tag{1.2}$$

Theorem 1.1. *With above set-up, for a generic X_0,*
(1) c_0 is an immersion , i.e. there exists a bundle, called normal bundle,

$$N_{c_0/X_0}$$

over $\mathbf{P}^1$ uniquely determined by c_0 such that the sequence

$$0 \quad \to \quad T_{\mathbf{P}^1} \quad \stackrel{(c_0)_*}{\to} \quad c_0^*(T_{X_0}) \quad \to \quad N_{c_0/X_0} \quad \to \quad 0,$$

is exact,
(2) the normal bundle satisfies

$$H^1(N_{c_0/X_0}) = 0. \tag{1.3}$$

Corollary 1.2. *Let $X_0 \subset \mathbf{P}^4$ be a generic quintic hypersurface, and c_0 as in theorem 1.1. Then the bundle N_{c_0/X_0} as in theorem 1.1 has an isomorphism*

$$N_{c_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

Proof. of corollary 1.2 following from theorem 1.1: Notice

$$deg(N_{c_0/X_0}) = deg(c_0^*(T_{X_0})) - deg(T_{\mathbf{P}^1}) = -2.$$

It is well-known that the bundle can be split into,

$$N_{c_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(k) \oplus \mathcal{O}_{\mathbf{P}^1}(-k-2) \tag{1.4}$$

where $k \geq -1$ is an integer. By Serre duality

$$\begin{aligned} H^1(N_{c_0/X_0}) &\simeq H^0((N_{c_0/X_0})^* \otimes \omega_{\mathbf{P}^1}) \\ &\simeq H^0(\mathcal{O}_{\mathbf{P}^1}(-2-k) \oplus \mathcal{O}_{\mathbf{P}^1}(k)). \end{aligned} \tag{1.5}$$

By theorem 1.1, $H^1(N_{c_0/X_0}) = 0$. Hence $-1 \leq k \leq -1$. Therefore $k = -1$.
$\square$

1.1 Outline of the proof

The cohomological statement of theorem 1.1 is equivalent to a property of the incidence scheme

$$\Gamma_{X_0} = \{\textit{birational to its image maps } c : \mathbf{P}^1 \to X_0\} \tag{1.6}$$

of rational maps to rational curves of a fixed degree on generic quintic threefold X_0— (1) Γ_{X_0} is reduced, (2) it has the expected dimension. The set of defining equations of this scheme are pretty easy to obtain (see [6]). This property, which is determined by the Jacobian matrix of this set of defining equations therefore is another expression of theorem 1.1. Clemens proved that there are components of Γ_{X_0} at whose generic points the Jacobian matrix has full rank ([3], section 1). But the method can't be used on all components. In this paper, we prove it for all components. Our general idea of using Jacobian matrices is similar to Clemens', but the detailed steps and the technique are different. We

(I) replace the single quintic X_0 by a generic two parameter family $\mathbb{L}$ of quintics.

(II) then show that the Jacobian matrix for the projection $P(\Gamma_{\mathbb{L}})$ at a generic point has full rank.

Therefore $P(\Gamma_{\mathbb{L}})$ is reduced with the expected dimension. Then it follows that the incidence scheme Γ_{X_0} is also reduced with the expected dimension (see [9] for the details).

By switching X_0 to $\mathbb{L}$, we obtain two free parameters for the incidence scheme $P(\Gamma_{\mathbb{L}})$ that come from the deformation of the quintic f_0, while the original component Γ_{X_0} has no free moduli parameters. The manipulation of two free parameters allows us to penetrate the Jacobian matrix. The following is the detailed sketch of the proof. For the parameter space of rational maps we use the linear model of moduli maps (used in (1.6)). In particular, we do not use a moduli space of rational maps. By the linear model we mean the affine space M,

$$M = \mathbb{C}^{5d+5} = (H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 5} \tag{1.7}$$

whose open subset parametrizes the set of non-constant regular maps

$$\mathbf{P}^1 \to \mathbf{P}^4$$

whose push-forward cycles have degree d. Let M_d be the subset that consists of all generically one-to-one (to its image) maps c whose images $c_*(\mathbf{P}^1)$ have degree d. Let

$S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5)))$ be the space of all quintics. Let $\mathbb{L} \subset S$ be an open set of the plane spanned by quintics f_0, f_1, f_2. Let

$$\Gamma_{\mathbb{L}} \ni (c_0, [f_0])$$

be an irreducible component of the incidence scheme

$$\{(c, [f]) \subset M_d \times \mathbb{L} : c^*(f) = 0\} \tag{1.8}$$

that is onto $\mathbb{L}$, where $[f_0]$ denotes the image of f_0 under the map

$$H^0(\mathcal{O}_{\mathbf{P}^4}(5)) - \{0\} \quad \to \quad S.$$

We assume $\Gamma_{\mathbb{L}}$ exists. Let P be the projection

$$\Gamma_{\mathbb{L}} \to M.$$

The idea of the proof is to show that the projection,

$$P(\Gamma_{\mathbb{L}}) \subset M$$

is a reduced, irreducible quasi-affine scheme of dimension 6. The method is straightforward to show its defining polynomials at a generic point have non-degenerate Jacobian matrix (by that we mean it has full rank). See definition 1.8 below for a precise definition of a Jacobian matrix. All differentials and partial derivatives used throughout the paper are in algebraic sense, i.e. defined as in [7] (because all functions are holomorphic). In the following we describe its defining polynomials and a differential form representing the Jacobian matrix. Choose generic $5d + 1$ distinct points $t_i \in \mathbf{P}^1$ (generic in $Sym^{5d+1}(\mathbf{P}^1)$). Throughout the paper, unless specified otherwise, we'll use t_i to denote a complex number which is a point in an affine open set $\mathbb{C} \subset \mathbf{P}^1$. Next we consider differential 1-forms ϕ_i on M:

$$\phi_i = d \begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} \tag{1.9}$$

for $i = 3, \cdots, 5d + 1$, and variable $c \in M$, where $| \cdot |$ denotes the determinant of a matrix. Notice ϕ_i are uniquely defined provided the quintics f_i are in an affine open set of S, and $t_i \in \mathbb{C}$ as chosen. Let

$$\omega(\mathbb{L}, \mathbf{t}) = \wedge_{i=3}^{5d+1} \phi_i \in H^0(\Omega_M^{5d-1}) \tag{1.10}$$

be the $5d - 1$-form. This $\omega(\mathbb{L}, \mathbf{t})$ is just a collection of all maximal minors of the Jacobian matrix of defining polynomials

$$\begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} \tag{1.11}$$

for the scheme $P(\Gamma_{\mathbb{L}})$, where $\mathbf{t} = (t_1, \cdots, t_{5d+1})$.

The following proposition asserts the non-degeneracy of the Jacobian matrix of the defining equations of $P(\Gamma_{\mathbb{L}})$.

PROPOSITION 1.3. *The form $\omega(\mathbb{L}, \mathbf{t})$ is not identically zero on $P(\Gamma_{\mathbb{L}})$.*

Then non-degeneracy of the Jacobian matrix means

PROPOSITION 1.4. *If $\omega(\mathbb{L}, \mathbf{t})$ is non-zero on $P(\Gamma_{\mathbb{L}})$, the Zariski tangent space of $P(\Gamma_{\mathbb{L}})$ at a generic maximal point must be*

$$dim(M) - deg(\omega(\mathbb{L}, \mathbf{t})) \tag{1.12}$$

The cohomological statement in theorem 1.1 follows immediately from the propositions on the incidence scheme above. See [9] for this step.

PROPOSITION 1.5. *If propositions 1.3, 1.4 are true, theorem 1.1 is true.*

Proposition (1.3) is the central part of the proof. It is a consequence of the study of a Jacobian matrix $\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t})$ of a large size $(5d+5) \times (5d+5)$, where C_M stands for local coordinates' system of the space M. In [9], we used the successive blow-ups to study the matrix $\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t})$ around a very degenerate point on $P(\Gamma_{\mathbb{L}})$. In this paper, we avoid the successive blow-ups by directly studying a generic point of $P(\Gamma_{\mathbb{L}})$.[1].

This can be done through a trick. Let us refer it as a "break-up trick". This is the process of a sequence of specializations. Roughly speaking, we compound the process of breaking up a whole matrix to block matrices, then manipulate the set, $C_M, f_i, \mathbf{t}$ and the base point $c_g \in P(\Gamma_{\mathbb{L}})$ to have computable block matrices. The trick is that we also need to break

$$C_M, f_i, \mathbf{t}, c_g$$

to study each block and there is no unified

$$C_M, f_i, \mathbf{t}, c_g$$

(generic in some sense) for all block matrices. But in the end all broken pieces with special sets of $C_M, f_i, \mathbf{t}, c_g$ must be chosen to coincide at the same generic

$$C_M, f_0, f_1, f_2, \mathbf{t}, c_g.$$

So specializations must **NOT** be applied to the entire matrix $\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t})$, but they are applied to some block matrices separately.

Let's give a detailed description of it in the following. It suffices to prove the proposition 1.3 for a specific $\mathbb{L}$. Thus we choose

$$f_0 = generic, f_1 = z_2 z_3 z_4 q, f_2 = z_0 z_1 z_2 z_3 z_4 \tag{1.13}$$

where $z_0, \cdots, z_4$ are homogeneous coordinates of $\mathbf{P}^4$, and q is a generic quadratic, homogeneous polynomial in $z_0, \cdots, z_4$. First we write down the differential one form $\phi_i, i = 3, \cdots, 5d+1$(expand it using Leibniz rule in differential):

$$\phi_i = d(g_i(c)) + \sum_{l=0,j=1}^{l=2,j=2} h^i_{lj} df_l(c(t_j)). \tag{1.14}$$

[1]But both methods rely on an algebro-geometric process, "specialization"

where g_i is a linear combination of $df_0(c(t_i)), df_1(c(t_i)), df_2(c(t_i))$. Then it suffices to show the polynomials

$$g_i(c) - g_i(c_g), f_l(c(t_j)) - f_l(c_g(t_j)), i = 3, \cdots, 5d+1, j = 1, 2, l = 0, 1, 2$$

at a generic point c_g of $P(\Gamma_{\mathbb{L}})$ form a regular system of parameters for the local ring $\mathcal{O}_{c_g,M}$ of M. This is the same to show the $(5d+5) \times (5d+5)$ Jacobian matrix $\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t})$

$$\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t}) = \frac{\partial(g_3, \cdots, g_{5d+1}, f_0(c(t_1)), \cdots, f_2(c(t_2)))}{\partial C_M} \tag{1.15}$$

is non-degenerate at c_g. The following is the trick mentioned above. For a generic f_0, q and a GENERIC $c_g \in U_{\mathbb{L}}$, we can choose a special C_M and $\mathbf{t}$ denoted by $C'_M, \mathbf{t}'$ such that,

$$\mathcal{A}(C'_M, f_0, f_1, f_2, \mathbf{t}')|_{c_g} \overset{Row}{\sim} \begin{pmatrix} I & 0 \\ 0 & Jac(C'_M, c_g) \end{pmatrix} \tag{1.16}$$

where I is the identity matrix of size $(5d-2) \times (5d-2)$ and $Jac(C'_M, c_g)$ is a 7×7 matrix (this is the break-up of the matrix and it is done in section 3, the step of choosing C_M)[2]. This $Jac(C'_M, c_g)$ is the most difficult part in $A(C_M, f_0, f_1, f_2, \mathbf{t})$. Next to penetrate the 7×7 matrix, $Ja(C'_M, c_g)$, we use the "break-up trick" to break the set of $C_M, \mathbb{L}, \mathbf{t}, c_g$, i.e. we choose a special $c_g^1 \in P(\Gamma_{\mathbb{L}})$ and another coordinates C''_M to show that $Jac(C''_M, c_g^1)$ is non-degenerate. Thus $Ja(C'_M, c_g)$ is non-degenerate. The trick is that those special $c_g 1, C''_M$ fail the formula (1.15), therefore should be avoided at the first place(there will be a couple of more similar break-ups of $Jac(C''_M, c_g^1)$ later). Therefore $\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t})$ is non-degenerate for any C_M, and generic $f_0, f_1, f_2, \mathbf{t}$.

1.2 Technical notations

In this section, we collect all technical notations and definitions used in this paper. Some of them may already be defined before.

Notations:
(1) S denotes the space all quintics, i.e. $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5)))$.
Let $[f]$ denote the image of f under the map

$$H^0(\mathcal{O}_{\mathbf{P}^4}(5)) - \{0\} \quad \rightarrow \quad S.$$

(2) Let

$$M$$

be

$$\mathbb{C}^{5d+5} \simeq (H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 5}$$

and M_d be the subset that parametrizes all birational-to-its-image maps

$$\mathbf{P}^1 \rightarrow \mathbf{P}^4$$

[2] This break-up requires that c_g is birational to its image. If c_g is a multiple cover map, this break-up does not hold.

whose push-forward cycles have degree d.

(3) Throughout the paper, if

$$c : \mathbf{P}^1 \to \mathbf{P}^4,$$

is regular, $c^*(\sigma)$ denotes the pull-back section of section σ of some bundle over $\mathbf{P}^4$. The vector bundles will not always be specified, but they are apparent in the context.

(4) Let Y be a scheme, $y \in Y$ be a closed point, $Z \subset Y$ be a subscheme (open or closed) and $\mathcal{M}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-module. Then $\mathcal{O}_{y,Y}$ denotes the local ring, Ω_Y denotes the sheaf of differentials, $\mathcal{M}|_{(Z)}$ denotes the inverse sheaf module $i^*(\mathcal{M})$ where $i : Z \hookrightarrow Y$ is the embedding. We call $\mathcal{M}|_{(Z)}$ the restriction of $\mathcal{M}$ to Z. $\mathcal{M}|_Z$ denotes the localization of $\mathcal{M}$ at Z, which is a $\mathcal{O}_{Z,Y}$ module. Thus

$$\mathcal{M}|_{(\{y\})} = \mathcal{M}|_Z \otimes k(y),$$

where $k(y)$ is the residue field of the maximal point $\{y\}$.

If Y is quasi-affine scheme, $\mathcal{O}(Y)$ denotes the ring of regular functions on Y.

(5) If Y is a scheme, $|Y|$ denotes the induced reduced scheme of Y.

Definition 1.6. *Let Γ be an irreducible component of the incidence scheme*

$$\{(c, f) \subset M_d \times \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5))) : c^*(f) = 0\} \tag{1.17}$$

that dominates $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5)))$. Let $(c_0, [f_0]) \in \Gamma$ be a generic point. Throughout the paper we assume that such a Γ exists.

Remark: The existence of such a Γ is equivalent to the assumption of theorem 1.1: X_0 is generic.

Definition 1.7. *Let $f_1, f_2 \in H^0(\mathcal{O}_{\mathbf{P}^4}(5))$ be two quintics different from f_0. Let $\mathbb{L}$ be an open set of the plane in*

$$\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5)))$$

spanned by $[f_0], [f_1], [f_2]$ and centered around $[f_0]$.

Let

$$\Gamma_{\mathbb{L}} = \Gamma \cap (M \times \mathbb{L}) \tag{1.18}$$

be an irreducible component of the restriction of Γ to $M \times \mathbb{L}$ such that it is onto $\mathbb{L}$, and

$$\Gamma_{f_0},\ for\ generic\ f_0 \in \mathbb{L} \tag{1.19}$$

is an irreducible component of

$$P(\Gamma \cap (M \times \{[f_0]\}))$$

where P is the projection to M.

Definition 1.8. *Let $\Delta^n \subset \mathbf{C}^n$ be an analytic open set with coordinates $x_1, \cdots, x_n$, Let $f_1, \cdots, f_m$ be holomorphic functions on Δ. For any positive integers $m' \leq m, n' \leq n$ and a point $p \in \Delta^n$, we define*

$$
(1.20) \qquad \frac{\partial(f_1,f_2,\cdots,f_{m'})}{\partial(x_1,x_2,\cdots,x_{n'})}|_p = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_{n'}} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_{n'}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_{m'}}{\partial x_1} & \frac{\partial f_{m'}}{\partial x_2} & \cdots & \frac{\partial f_{m'}}{\partial x_{n'}} \end{pmatrix} |_p.
$$

to be the Jacobian matrix of functions $f_1, \cdots, f_{m'}$ in $x_1, \cdots, x_{n'}$. If $n' = n \geq m' = m$, the differential form $\wedge_{i=1}^m df_i$ is just the collection of all $m \times m$ minors of the Jacobian matrix.

This Jacobian matrix is just the differential of the composition map

$$
(1.21) \qquad \begin{array}{c} (x_1, \cdots, x_{n'}) \\ \downarrow \\ (x_1, \cdots, x_{n'}, x_{n'+1}, \cdots, x_n) \\ \downarrow \\ (f_1, \cdots, f_m) \\ \downarrow \\ (f_1, \cdots, f_{m'}), \end{array}
$$

where the first map is defined by the embedding via the coordinates of p.

This definition depends on all coordinates $x_1, \cdots, x_n$ and it is crucial. One of main difficulties of this paper is to search for such coordinates that would make Jacobian matrices simpler.

In section 2, we prove that original Clemens' conjecture follows from theorem 1.1. In section 3, we prepare the analytic coordinates of M for the computation. In section 4, we use the sheaf of differentials to show the non-vanishing property of $5d$-1-form $\omega(M, \mathbf{t})$ on the scheme $\mathbf{P}(\Gamma_{\mathbb{L}})$. This is the central section of the paper. It leads the proof of propositions 1.3, 1.4. Section 5 collects two known examples which emphasize on the singular rational curves.

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2 Clemens' conjecture

Rational curves on hypersurfaces have been great interests for many years in algebraic geometry. The Clemens' conjecture sits in the center of many major problems in this area. In [2], its original 1986 statement, Clemens proposed:

"(1) the generic quintic threefold V admits only finitely many rational curves of each degree.

(2) Each rational curve is a smoothly embedded $\mathbf{P}^1$ with normal bundle

$$
(2.1) \qquad \mathcal{O}_{\mathbf{P}^1}(-1) \oplus O_{\mathbf{P}^1}(-1).
$$

(3) All the rational curves on V are mutually disjoint. The number of rational curves of degree d on V is

$$(interesting\ number) \cdot 5^3 \cdot d." \tag{2.2}$$

During the last thirty years, there are many articles on the conjecture. The most of them followed the early idea of Katz ([6]) to show that there is only one irreducible component of the incidence scheme, containing a smooth rational curve and dominating the space of quintics. In 1995, Vainsencher found the degree 5, 6-nodal rational curves in the generic quintic threefolds ([8]). This partially disproved part (2) in the Clemens' conjecture and leave the part (1) unanswered. At the meantime Mirror symmetry came to the stage to redefine the approach in part (3). Based on Vainsencher's result, in 1999, motivated by the Gromov-Witten invariants in the mirror symmetry, Cox and Katz modified the Clemens' original conjecture to the most current form ([4]):

" Let $V \subset \mathbf{P}^4$ be a generic quintic threefold. Then for each degree $d \geq 1$, we have

(i) There are only finitely many irreducible rational curves $C \subset V$ of degree d.

(ii) These curves, as we vary over all degree, are disjoint from each other.

(iii) If $c : \mathbf{P}^1 \to C$ is the normalization of an irreducible rational curve C, then the normal bundle has isomorphism

$$N_{c/V} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)."$$

Remark. Cox and Katz's conjecture (iii) should be understood as in two steps. First $N_{c/V}$ must be a locally free sheaf, secondly

$$N_{c/V} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

We proved the first by showing that c_0 is an immersion.

The conjecture is proved to be correct for $d \leq 9$ by the work of Katz ([6]), Johnsen and Kleiman ([5]), and Cox and Katz ([4]), etc.

2.1 A proof of Clemens' conjecture

Clemens' conjecture follows from Theorem 1.1 and corollary 1.2 because the corollary below

Corollary 2.1. *Let $X_0 \subset \mathbf{P}^4$ be a generic quintic threefold. Then for each degree $d \geq 1$, we have*

(i) there are only finitely many irreducible rational curves $C_0 \subset X_0$ of degree d.

(ii) Each rational curve in (i) is an immersed rational curve with normal bundle

$$N_{c_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

By "immersed rational curve" we mean that the normalization map is an immersion.

Proof. of corollary 2.1 following from theorem 1.1 and corollary 1.2: The existence of rational curves on a generic quintic was proved in [3], [6]. So it suffices to prove the finiteness. Part (i) follows from part (ii). So let's prove part (ii). Let C_0 be an

irreducible rational curve of degree d on X_0. Then we take a normalization of C_0, and denote it by $c_0 : \mathbf{P}^1 \to X_0$. Since X_0 is generic, we have the set-up for corollary 1.2. Applying corollary 1.2, we obtain part (ii). $\square$

Corollary 2.1 proves the modified Clemens' conjecture, namely parts (i) and (ii) of Cox and Katz's statements. Clemens' original conjecture must be modified in the light of Vainsencher's result.

3 Space of rational curves, M

The basis of this paper is the linear model of stable moduli, which begins with the projectivization $\mathbf{P}(M)$. The space M is an affine space $\mathbb{C}^{5d+5}$, therefore is very simple. But we are interested in some subschemes which are not trivial at all. Our idea is to introduce various analytic coordinates of each copy $\mathbb{C}^{d+1}$ in $\mathbb{C}^{5d+5}$. The purpose of these coordinates is to provide various parameters for the local ring $\mathcal{O}_{c,M}$ so that the Jacobian matrices under these coordinates are either diagonal or triangular. In this section we introduce a couple of coordinates systems C'_M, C''_M that will be used for our "break-up trick".

However readers may skip this section because without section 4 technical preparation here may seem to be aimless.

Let $c_g = (c_g^0, \cdots, c_g^4) \in M_d$ with

$$c_g^i \in H^0(\mathcal{O}_{\mathbf{P}^1}(d)) - \{0\}, i = 0, \cdots, 4.$$

We may assume $t \in \mathbb{C} \subset \mathbf{P}^1$. Because $c_g \in M_d$, we assume $c_g^i(t) = 0, i = 0, \cdots, 4$ have $5d$ distinct zeros

$$\tilde{\theta}_i^j, for\ i \leq 4.$$

Then each component, $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ of

$$M = H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 5}$$

has local analytic "polar" coordinates

$$r_i, \theta_i^j, j = 1, \cdots, d, for\ r_i \neq 0 \tag{3.1}$$

(for each $i = 0, 1, 2, 3, 4$) around c_g such that

$$c^i(t) = r_i \prod_{j=1}^{d} (t - \theta_i^j). \tag{3.2}$$

Let coordinates values for c_g be

$$r_l = y_l, \theta_i^j = \tilde{\theta}_i^j, l = 0, \cdots, 4, i = 0, \cdots, 4, j = 1, \cdots, d.$$

Let q be a generic, homogeneous quadratic polynomial in $z_0, \cdots, z_4$.

Let

$$h(c,t) = \delta_1 q(c(t)) + \delta_2 c_3(t) c_4(t). \tag{3.3}$$

for $c \in M$, where $\delta_i, i = 1,2$ are two none zero complex numbers. Let $\beta_1, \cdots, \beta_{2d}$ be the zeros of $h(c_g, t) = 0$. Also let

$$h(c,t) = \xi \prod_{i=1}^{2d} (t - \epsilon_i).$$

It is clear that

$$\begin{gathered} \xi = \delta_1 q(r_0, r_1, r_2, r_3, r_4) + \delta_2 r_3 r_4, \ and \\ \epsilon_i \ are \ analytic \ functions \ of \ c. \end{gathered}$$

Let the corresponding value of ξ at c_g be ξ^0. By the genericity of q, we may assume $\beta_i, i = 1, \cdots, 2d$ are distinct and non-zeros . Furthermore we assume β_i are distinct for $q = z_1 z_2$ and generic δ_i.

Proposition 3.1. *Let $U_{c_g} \subset M$ be an analytic neighborhood of c_g.*
(a) Let

$$\varrho : U_{c_g} \quad \rightarrow \quad \mathbb{C}^{5d+5} \tag{3.4}$$

be a regular map that is defined by

$$\begin{gathered} (\theta_0^1, \cdots, \theta_4^d, r_0, r_1, r_2, r_3, r_4) \\ \downarrow \varrho \\ (\theta_0^1, \cdots, \theta_2^d, \epsilon_1, \cdots, \epsilon_{2d}, r_0, \cdots, r_3, \xi). \end{gathered} \tag{3.5}$$

Then ϱ is an isomorphism to its image.
(b) Let

$$\varrho' : U_{c_g} \quad \rightarrow \quad \mathbb{C}^{5d+5} \tag{3.6}$$

be a regular map that is defined by

$$\begin{gathered} (\theta_0^1, \cdots, \theta_4^d, r_0, r_1, r_2, r_3, r_4) \\ \downarrow \varrho' \\ (\theta_0^1, \cdots, \theta_2^d, \epsilon_1, \cdots, \epsilon_{2d}, r_0, \cdots, r_3, r_4). \end{gathered} \tag{3.7}$$

Then ϱ' is an isomorphism to its image.

Proof. It suffices to prove the differential of ϱ at c_g is an isomorphism for a SPECIFIC q. So we assume that

$$\delta_1 = \delta_2 = 1, q = z_1 z_2$$

This is a straightforward calculation of the Jacobian determinant of ϱ. We may still assume that $\beta_i, i = 1, \cdots, 2d$ are distinct. Using the composition of two isomorphisms, we obtain that the Jacobian determinant

$$det(\frac{\partial(\tilde\theta_0^1, \cdots, \tilde\theta_2^d, y_0, \cdots, y_3, \xi^0, \beta_1, \cdots, \beta_{2d})}{\partial(\theta_0^1, \cdots, \theta_2^d, r_0, r_1, r_2, r_3, r_4, \theta_3^1, \cdots, \theta_4^d)}) \tag{3.8}$$

is equal to

$$a \cdot \frac{\partial \xi}{\partial r_4}|_{c_g} \cdot J \tag{3.9}$$

where a is some non-zero number, $\frac{\partial \xi}{\partial r_4}|_{c_g}$ is also non-zero and J is another Jacobian

$$J = \begin{vmatrix} \frac{\partial h(c,\beta_1)}{\partial \theta_3^1} & \cdots & \frac{\partial h(c,\beta_1)}{\partial \theta_4^d} \\ \vdots & \vdots & \vdots \\ \frac{\partial h(c,\beta_{2d})}{\partial \theta_3^1} & \cdots & \frac{\partial h(c,\beta_{2d})}{\partial \theta_4^d} \end{vmatrix}_{\tilde{c}_2} \tag{3.10}$$

Let $T_i, i = 0, d$ be the determinant

$$\begin{vmatrix} \beta_{i+1} & \cdots & \beta_{i+1}^d \\ \vdots & \vdots & \vdots \\ \beta_{i+d} & \cdots & \beta_{i+d}^d \end{vmatrix}. \tag{3.11}$$

Then we compute the determinant to have

$$J = (-1)^d T_0 T_d \prod_{i=1}^{d} (c_g^3(\beta_{d+i}) c_g^4(\beta_i) - c_g^3(\beta_i) c_g^4(\beta_{d+i})). \tag{3.12}$$

Since β_i are distinct and non-zeros,

$$T_0 \neq 0, T_d \neq 0.$$

Since $\frac{c_g^3(t)}{c_g^4(t)}$ is a non-constant rational function and

$$deg(c_g^3(t)) = deg(c_g^4(t)) = d$$

then we can always arrange the indexes of β_i such that each number

$$(\frac{c_g^3(\beta_{d+i})}{c_g^4(\beta_{d+i})} - \frac{c_g^3(\beta_i)}{c_g^4(\beta_i)}) \tag{3.13}$$

is not zero. Hence

$$\prod_{i=1}^{d} (c_g^3(\beta_{d+i}) c_g^4(\beta_i) - c_g^3(\beta_i) c_g^4(\beta_{d+i})) \neq 0.$$

Thus J is non-zero. Therefore

$$det(\frac{\partial(\tilde{\theta}_0^1, \cdots, \tilde{\theta}_2^d, y_0, \cdots, y_3, \xi^0, \beta_1, \cdots, \beta_{2d})}{\partial(\theta_0^1, \cdots, \theta_2^d, r_0, r_1, r_2, r_3, r_4, \theta_3^1, \cdots, \theta_4^d)}) \neq 0 \tag{3.14}$$

The proof of part (b) is the same as for part (a). We complete the proof.
□

Definition 3.2. *By proposition 4.1, both*

$$\theta_0^1, \cdots, \theta_2^d, r_0, \cdots, r_3, \xi, \epsilon_1, \cdots, \epsilon_{2d} \tag{3.15}$$

and

$$\theta_0^1, \cdots, \theta_2^d, r_0, \cdots, r_4, \epsilon_1, \cdots, \epsilon_{2d} \tag{3.16}$$

are local analytic coordinates of M around c_g, and c_g corresponds to the coordinate values

$$\begin{gathered} \theta_i^j = \tilde{\theta}_i^j, i = 0, 1, 2, j = 1, \cdots, d \\ r_l = y_l \neq 0, l = 0, \cdots, 4 \\ \epsilon_i = \beta_i, i = 1, \cdots, 2d \end{gathered} \tag{3.17}$$

and ξ^0.

For the simplicity, we refer the first coordinates' system

$$\theta_0^1, \cdots, \theta_2^d, r_0, \cdots, r_3, \xi, \epsilon_1, \cdots, \epsilon_{2d} \tag{3.18}$$

as C'_M and the second one

$$\theta_0^1, \cdots, \theta_2^d, r_0, \cdots, r_4, \epsilon_1, \cdots, \epsilon_{2d} \tag{3.19}$$

as C''_M.

The following lemma is also a local expression for the calculation later. Choose homogeneous coordinates $[z_0, \cdots, z_4]$ for $\mathbf{P}^4$. Let

$$f_3 = z_0 z_1 z_2 (\delta_1 q + \delta_2 z_3 z_4). \tag{3.20}$$

where δ_i are two non-zero complex numbers, and q is a generic, quadratic homogeneous polynomial in $z_0, \cdots, z_4$. Let $c_g \in M_d$ as above.

$$f_3(c_g(t)) \neq 0.$$

We denote the zeros of $c_g^i(t) = 0$ by $\tilde{\theta}_i^j$ and zeros of

$$(\delta_1 q + \delta_2 z_3 z_4|_{c_g(t)}) = 0 \tag{3.21}$$

by $\beta_i, i = 1, \cdots, 2d$. We assume $\tilde{\theta}_i^j, i = 0, \cdots, 4, j = 1, \cdots, d$ are distinct, and $\beta_i, i = 1, \cdots, 2d$ are also distinct.

Lemma 3.3. *Recall in definition 4.2,*

$$\theta_0^1, \cdots, \theta_2^d, r_0, \cdots, r_3, \xi, \epsilon_1, \cdots, \epsilon_{2d}$$

are analytic coordinates of M around the point c_g.

Then

(a) the Jacobian matrix

$$
\begin{array}{c}
J(\tilde{c}_2) \\
\| \\
\begin{pmatrix}
\frac{\partial f_3(c_g(\tilde{\theta}_0^1))}{\partial \theta_0^1} & \cdots & \frac{\partial f_3(c_g(\tilde{\theta}_0^1))}{\partial \theta_2^d} & \frac{\partial f_3(c_g(\tilde{\theta}_0^1))}{\partial \epsilon_1} & \cdots & \frac{\partial f_3(c_g(\tilde{\theta}_0^1))}{\partial \epsilon_{2d}} \\
\frac{\partial f_3(c_g(\tilde{\theta}_0^2))}{\partial \theta_0^1} & \cdots & \frac{\partial f_3(c_g(\tilde{\theta}_0^2))}{\partial \theta_2^d} & \frac{\partial f_3(c_g(\tilde{\theta}_0^1))}{\partial \epsilon_1} & \cdots & \frac{\partial f_3(c_g(\tilde{\theta}_0^2)}{\partial \epsilon_{2d}} \\
\frac{\partial f_3(c_g(\tilde{\theta}_0^3))}{\partial \theta_0^1} & \cdots & \frac{\partial f_3 c_g(\tilde{\theta}_0^3))}{\partial \theta_2^d} & \frac{\partial f_3(c_g(\tilde{\theta}_0^3))}{\partial \epsilon_1} & \cdots & \frac{\partial f_3(c_g(\tilde{\theta}_0^3))}{\partial \epsilon_{2d}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_3(c_g(\tilde{\theta}_4^d))}{\partial \theta_0^1} & \cdots & \frac{\partial f_3(c_g(\tilde{\theta}_4^d))}{\partial \theta_2^d} & \frac{\partial f_3(c_g(\tilde{\theta}_4^d))}{\partial \epsilon_1} & \cdots & \frac{\partial f_3(c_g(\tilde{\theta}_0^d))}{\partial \epsilon_{2d}}
\end{pmatrix}
\end{array}
\tag{3.22}
$$

is equal to a diagonal matrix D whose diagonal entries are

$$
\frac{\partial f_3(c_g(\tilde{\theta}_0^1))}{\partial \theta_0^1}, \cdots, \frac{\partial f_3(c_g(\tilde{\theta}_2^d)}{\partial \theta_2^d}, \frac{\partial f_3(c_g(\tilde{\theta}_3^1))}{\partial \epsilon_1}, \cdots, \frac{\partial f_3(c_g(\tilde{\theta}_4^d))}{\partial \epsilon_{2d}}
\tag{3.23}
$$

which are all non-zeros.

(b) For $i = 0, \cdots, 4$, $j = 1, \cdots, d$, $l = 0, \cdots, 3$

$$
\frac{\partial f_3(c_g(\tilde{\theta}_i^j))}{\partial r_l} = \frac{\partial f_3(\tilde{c}_2(t_i))}{\partial \xi} = 0.
$$

Proof. Note $\tilde{\theta}_i^j, i = 0, \cdots, 4, j = 1, \cdots, d$ are distinct and $\beta_i, i = 0, \cdots, 2d$ are also distinct. Thus the coordinates in definition 4.2 exist. We can rewrite

$$
f_3(c(t)) = y \prod_{j=1}^{5d} (t - \alpha_j)
\tag{3.24}
$$

around $\tilde{c}_2$. Then y, α_j are all functions of the analytic coordinates in definition 4.2,

$$
\theta_i^j, \epsilon_1, \cdots, \epsilon_{2d}, y_l.
\tag{3.25}
$$

More specifically $\theta_i^j, i \leq 2$ are exactly the $3d$ roots, $\alpha_i, i = 1, \cdots, 3d$ and $\epsilon_i, i = 1, \cdots, 2d$ are α_{3d+i}, and y is an analytic function of $y_l, l = 0, \cdots, 4$. Hence

$$
f_3(c(t)) = r_0 r_1 r_2 \xi \prod_{i=0, j=1, l=1}^{i=2, j=d, l=2d} (t - \theta_i^j)(t - \epsilon_l).
\tag{3.26}
$$

Both parts of lemma 4.3 follow from the expression (4.26). We complete the proof. ∎

4 Differential sheaf

In this section, we prove theorem 1.1, i.e.

$$
H^1(N_{c_0/X_0}) = 0
\tag{4.1}
$$

at generic $(c_0, [f_0]) \in \Gamma$.

4.1 Non-vanishing $5d-1$-form $\omega(\mathbb{L}, \mathbf{t})$

LEMMA 4.1.

The $5d$-1 form $\omega(\mathbb{L}, \mathbf{t})$ defined in (1.10) is a non-zero form when it is evaluated at generic points of $P(\Gamma_{\mathbb{L}})$, i.e. the reduction $\bar{\omega}(\mathbb{L}, \mathbf{t})$ in the module,

$$H^0(\Omega_M \otimes \mathcal{O}_{P(\Gamma_{\mathbb{L}})})$$

is non zero.

This lemma is proposition 1.3 in the introduction.

It suffices to prove lemma 4.1 for special choices of f_0, f_1, f_2 and $t_1, \cdots, t_{5d+1}$. So let $z_0, z_1, \cdots, z_4$ be general homogeneous coordinates of $\mathbf{P}^4$. Let

$$f_2 = z_0 z_1 z_2 z_3 z_4.$$

Let

$$f_1 = z_0 z_1 z_2 q,$$

where q is a generic quadratic homogeneous polynomial in $z_0, \cdots, z_4$. Choose another generic f_0. Let

$$c_g \in P(\Gamma_{\mathbb{L}})$$

be a generic in $P(\Gamma_{\mathbb{L}})$. By the genericity of f_0, we may assume $c_g = (c_g^0, \cdots, c_g^4)$ such that $c_g^i \neq 0$ and $c_g^i(t) = 0, i = 0, \cdots, 4$ have $5d$ distinct zeros $\tilde{\theta}_i^j \in \mathbf{P}^1$. To choose $5d$ points t_i on $\mathbb{C} \subset \mathbf{P}^1$, we let

(1) t_1, t_2, t_{5d+1} be general among all (t_1, t_2, t_3) satisfying

$$\begin{vmatrix} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{vmatrix} = 0, \tag{4.2}$$

(2) $t_3, \cdots, t_{5d}$ be the $5d-2$ complex numbers

$$\begin{gathered} \tilde{\theta}_i^j, (i,j) \neq (0,1), (1,1), i \leq 2 \\ \beta_i, i = 1, \cdots, 2d \end{gathered} \tag{4.3}$$

where β_i are the zeros of

$$\delta_1 q(c_g(t)) + \delta_2 z_3 z_4|_{c(t)} = 0. \tag{4.4}$$

where

$$\begin{gathered} \delta_1 = \begin{vmatrix} f_0(c_g(t_1)) & f_2(c_g(t_1)) \\ f_0(c_g(t_2)) & f_2(c_g(t_2)) \end{vmatrix}, \\ \delta_2 = \begin{vmatrix} f_1(c_g(t_1)) & f_0(c_g(t_1)) \\ f_1(c_g(t_2)) & f_0(c_g(t_2)) \end{vmatrix}. \end{gathered} \tag{4.5}$$

To simply put it, $t_3, \cdots, t_{5d}$ are just the zeros of

$$\delta_1 f_1(c(t)) + \delta_2 f_2(c(t)) = 0. \tag{4.6}$$

excluding two zeros $\tilde{\theta}_0^1, \tilde{\theta}_1^1$.

We claim that

$$\delta_1 \neq 0, \quad \delta_2 \neq 0. \tag{4.7}$$

This is because c_g lies in a plane $\mathbb{L}$, but does not lie in the pencils $span(f_0, f_1), span(f_0, f_2)$. Thus

$$\{(f_0(c_g(t)), f_1(c_g(t)))\}_{t\in\mathbf{P}^1}$$

span $\mathbb{C}^2$. This implies

$$\begin{vmatrix} f_1(c_g(t_1)) & f_0(c_g(t_1)) \\ f_1(c_g(t_2)) & f_0(c_g(t_2)) \end{vmatrix} \neq 0$$

Similarly

$$\begin{vmatrix} f_0(c_g(t_1)) & f_2(c_g(t_1)) \\ f_0(c_g(t_2)) & f_2(c_g(t_2)) \end{vmatrix} \neq 0.$$

We expand $\phi_i, i = 3, \cdots, 5d$ to obtain that

$$\begin{aligned} \phi_i = & \begin{vmatrix} f_0(c(t_1)) & f_2(c(t_1)) \\ f_0(c(t_2)) & f_2(c(t_2)) \end{vmatrix} df_1(c(t_i)) + \begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{vmatrix} df_0(c(t_i)) \\ & + \begin{vmatrix} f_1(c(t_1)) & f_0(c(t_1)) \\ f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} df_2(c(t_i)) + \sum_{l=0,j=1}^{l=2,j=2} h_{lj}^i(c_g) df_l(c(t_j)) \end{aligned} \tag{4.8}$$

By the assumption for t_1, t_2,

$$\begin{vmatrix} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{vmatrix} = 0.$$

We obtain

$$\begin{aligned} \phi_i|_{c_g} &= \delta_1 df_1(c(t_i)) + \delta_2 df_2(c(t_i)) + \sum_{l=0,j=1}^{l=2,j=2} h_{lj}^i(c_g) df_l(c(t_j)) \\ &= df_3(c(t_i)) + \sum_{l=0,j=1}^{l=2,j=2} h_{lj}^i(c_g) df_l(c(t_j)) \end{aligned} \tag{4.9}$$

where

$$f_3 = \delta_1 f_1 + \delta_2 f_2.$$

Notice $\delta_1 \neq 0 \neq \delta_2$.

To show lemma 4.1, it suffices to show the local holomorphic functions

$$\begin{gathered} f_3(c(t_3)), \cdots, f_3(c(t_{5d+1})), \\ f_0(c(t_1)), f_1(c(t_1)), f_2(c(t_1)), \\ f_0(c(t_2)), f_1(c(t_2)), f_2(c(t_2)). \end{gathered} \tag{4.10}$$

are the parameters of the local ring $\mathcal{O}_{c_g, M}$.

Let

$$\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t}') \tag{4.11}$$

be the Jacobian matrix of functions in (4.10) under an analytic coordinate's system C_M at c_g.

Then the lemma 4.1 follows from the following lemma

LEMMA 4.2. *The* $(5d+5) \times (5d+5)$ *matrix*

$$\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t}') \tag{4.12}$$

is non-degenerate.

Proof. of lemma 4.2: Let's choose C_M to be C'_M defined in 4.2. Recall C'_M has coordinates

$$\begin{gathered} \theta_i^j, i \leq 2 \\ \epsilon_i, i = 1, \cdots, 2d \\ r_0, \cdots, r_3, \xi \end{gathered} \tag{4.13}$$

The matrix $\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t}')$ is straightforward. But we would like to break it up to a block matrix

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix} \tag{4.14}$$

where $\mathcal{A}_{ij}$ are the following Jacobian matrices:

(a)

$$\mathcal{A}_{11} = \frac{\partial\Big(f_3(c(t_3)), f_3(c(t_4)), \cdots, f_3(c(t_{5d}))\Big)}{\partial(\theta_0^2, \cdots, \hat{\theta}_1^1, \cdots, \theta_2^d, \epsilon_1, \cdots, \epsilon_{2d})} \tag{4.15}$$

(b)

$$\begin{gathered} \mathcal{A}_{12} \\ \| \\ \frac{\partial\Big(f_3(c(t_{5d+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2))\Big)}{\partial(\theta_0^1, \theta_1^1, r_0, r_1, r_2, r_3, \xi)} \end{gathered} \tag{4.16}$$

(c)

$$\begin{gathered} \mathcal{A}_{21} \\ \| \\ \frac{\partial\Big(f_3(c(t_{5d+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2))\Big))}{\partial(\theta_0^2, \cdots, \hat{\theta}_1^1, \cdots, \theta_2^d, \epsilon_1, \cdots, \epsilon_{2d})} \end{gathered} \tag{4.17}$$

(d)

$$\begin{gathered} \mathcal{A}_{22} \\ \| \\ \frac{\partial\Big(f_3(c(t_{5d+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2))\Big)}{\partial(\theta_0^1, \theta_1^1, r_0, r_1, r_2, r_3, \xi))} \end{gathered} \tag{4.18}$$

Using lemma 3.3, part (a), $\mathcal{A}_{11}|_{c_g}$ is a non-zero diagonal matrix and

$$\mathcal{A}_{12}|_{c_g} = 0.$$

Therefore it suffices to show

$$det(\mathcal{A}_{22}) \neq 0. \tag{4.19}$$

Next we apply the "break-up trick", i.e. we'll change the parameters that determine the matrix, but the change will not effect its non-degeneracy [3]. First we change the coordinates to C''_M. More precisely we consider

$$\begin{array}{c} \mathcal{A}_{22}(C''_M) \\ \| \\ \frac{\partial\left(f_3(c(t_{5d+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2)) \right)}{\partial(\theta_0^1, \theta_1^1, r_0, r_1, r_2, r_3, r_4))} \end{array} \tag{4.20}$$

Denote the original $\mathcal{A}_{22}$ by $\mathcal{A}_{22}(C'_M)$. Because at c_g, the transformation of coordinates gives a relation

$$\mathcal{A}_{22}(C'_M) = \mathcal{A}_{22}(C'_M)D_7, \tag{4.21}$$

where D_7 is a non-degenerate 7×7 triangular matrix, it suffices to show

$$\mathcal{A}_{22}(C''_M)$$

is non-degenerate. Notice t_{5d+1} is generic on $\mathbf{P}^1$. The genericity of q makes curve in $\mathbb{C}^7$,

$$\left(\frac{\partial f_3(c(t_{5d+1}))}{\partial \theta_0^1}, \frac{\partial f_3(c(t_{5d+1}))}{\partial \theta_1^1}, \frac{\partial f_3(c(t_{5d+1}))}{\partial r_0}, \cdots, \frac{\partial f_3(c(t_{5d+1}))}{\partial r_4}\right) \tag{4.22}$$

span the entire space $\mathbb{C}^7$. This means the first row vector of

$$\mathcal{A}_{22}(C''_M)$$

is generic with respect to other 6 row vectors. Hence it suffices for us to show the Jacobian matrix

$$\begin{array}{c} \mathcal{B}(c_g) \\ \| \\ \frac{\partial(f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2)))}{\partial((\theta_0^1, \theta_1^1, r_1, r_2, r_3, r_4))} \end{array} \tag{4.23}$$

is non degenerate (the column of partial derivatives with respect to r_0 is eliminated). Now we use the "break-up trick" again. This time we change the point c_g. To show $\mathcal{B}(c_g)$ is non degenerate for a generic $c_g \in P(\Gamma_{\mathbb{L}})$, it suffices to show it is non-degenerate for a special $c_g \in P(\Gamma_{\mathbb{L}})$. To do that, we let $\mathbb{L}_1$ be an open set of pencil through f_0, f_2. $P(\Gamma_{\mathbb{L}_1})$ be as defined in lemmas 3.2, 3.3. We choose $P(\Gamma_{\mathbb{L}_1})$ to be irreducible, and to be contained in $P(\Gamma_{\mathbb{L}})$ for generic q (simultaneously). So a generic

[3] This is a trick because the change can't be made before the matrix is broken or reduced. For example the change of coordinate's system we are going to make below can't be applied to the entire matrix $\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t}')$.

point $c_g^1 = (c_1^0, \cdots, c_1^4) \in P(\Gamma_{\mathbb{L}_1})$ lies in M_d. Use the same notations for C'_M, C''_M values of c_g^1 as in definition 3.2. Because q is generic with respect to 1st, 2nd, 5th and 6th rows, two middle rows

$$\begin{gathered} \left(\frac{\partial f_1(c(t_1))}{\partial \theta_0^1}, \frac{\partial f_1(c(t_1))}{\partial \theta_1^1}, \frac{\partial f_1(c(t_1))}{\partial r_1}, \cdots, \frac{\partial f_1(c(t_1))}{\partial r_4}\right) \\ \left(\left(\frac{\partial f_1(c(t_2))}{\partial \theta_0^1}, \frac{\partial f_1(c(t_2))}{\partial \theta_1^1}, \frac{\partial f_1(c(t_2))}{\partial r_1}, \cdots, \frac{\partial f_1(c(t_2))}{\partial r_4}\right)\right. \end{gathered} \tag{4.24}$$

must be generic in $\mathbb{C}^6$ with respect to 1st, 2nd, 5th and 6th rows. Then we reduce $\mathcal{B}(c_g)$ to show

$$Jac(f_0, c_g^1) = \frac{\partial(f_3(c(t_1)), f_3(c(t_1)), f_0(c(t_1)), f_0(c(t_2)))}{\partial(\theta_0^1, r_2, r_3, r_4)} \tag{4.25}$$

is non-degenerate. Finally we write down the matrix $Jac(f_0, c_g^1)$,

$$\begin{gathered} Jac(f_0, c_g^1) \\ \| \\ \lambda \begin{pmatrix} \frac{1}{t_1 - \tilde{\theta}_0^1} & 1 & 1 & 1 \\ \frac{1}{t_2 - \tilde{\theta}_0^1} & 1 & 1 & 1 \\ \frac{\partial f_0(c_g^1(t_1))}{\partial \theta_0^1} & (z_2 \frac{\partial f_0}{\partial z_2})|_{c_g^1(t_1)} & (z_3 \frac{\partial f_0}{\partial z_3})|_{c_g^1(t_1)} & (z_4 \frac{\partial f_0}{\partial z_4})|_{c_g^1(t_1)} \\ \frac{\partial f_0(c_g^1(t_2))}{\partial \theta_0^1} & (z_2 \frac{\partial f_0}{\partial z_2})|_{c_g^1(t_2)} & (z_3 \frac{\partial f_0}{\partial z_3})|_{c_g^1(t_2)} & (z_4 \frac{\partial f_0}{\partial z_4})|_{c_g^1(t_2)} \end{pmatrix}, \end{gathered} \tag{4.26}$$

where λ is a non-zero complex number. We further compute to have

$$\begin{gathered} Jac(f_0, c_g^1) \\ \| \\ \lambda\left(\frac{1}{t_1 - \tilde{\theta}_0^1} - \frac{1}{t_2 - \tilde{\theta}_0^1}\right) \begin{pmatrix} 1 & 1 & 1 \\ (z_2 \frac{\partial f_0}{\partial z_2})|_{c_g^1(t_1)} & (z_3 \frac{\partial f_0}{\partial z_3})|_{c_g^1(t_1)} & (z_4 \frac{\partial f_0}{\partial z_4})|_{c_g^1(t_1)} \\ (z_2 \frac{\partial f_0}{\partial z_2})|_{c_g^1(t_2)} & (z_3 \frac{\partial f_0}{\partial z_3})|_{c_g^1(t_2)} & (z_4 \frac{\partial f_0}{\partial z_4})|_{c_g^1(t_2)} \end{pmatrix}. \end{gathered} \tag{4.27}$$

Since t_1, t_2 are only required to satisfy one equation (5.2), by the genericity of q, we may assume $(t_1, t_2) \in \mathbb{C}^2$ is generic. It suffices to prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ (z_2 \frac{\partial f_0}{\partial z_2})|_{c_2(t_1)} & (z_3 \frac{\partial f_0}{\partial z_3})|_{c_2(t_1)} & (z_4 \frac{\partial f_0}{\partial z_4})|_{c_2(t_1)} \\ (z_2 \frac{\partial f_0}{\partial z_2})|_{c_2(t_2)} & (z_3 \frac{\partial f_0}{\partial z_3})|_{c_2(t_2)} & (z_4 \frac{\partial f_0}{\partial z_4})|_{c_2(t_2)} \end{vmatrix} \neq 0$$

for any generic f_0 and c_2 that has no multiple zeros with coordinates planes $\{z_i = 0\}$.

Let Σ be an open subvariety

$$\{c \in M : zeros\ of\ c_i(t) = 0\ are\ distinct, i = 1, \cdots, 4\}.$$

Consider the family of rational maps

$$V_f = \{c \in \Sigma : Jac(f, c) = 0\}.$$

Notice by the definition V_f is a subvariety of Σ. Next we consider the fibre $V_{f_{Fe}}$ where f_{Fe} is the Fermat quintic

$$f_{Fe} = z_0^5 + \cdots + z_4^5.$$

It is obvious $V_{f_{Fe}}$ is empty. Hence V_f is empty for generic f. This shows that

$$\begin{vmatrix} 1 & 1 & 1 \\ (z_2 \frac{\partial f_0)}{\partial z_2})|_{c_2(t_1)} & (z_3 \frac{\partial f_0)}{\partial z_3})|_{c_2(t_1)} & (z_4 \frac{\partial f_0)}{\partial z_4})|_{c_2(t_1)} \\ (z_2 \frac{\partial f_0)}{\partial z_2})|_{c_2(t_2)} & (z_3 \frac{\partial f_0)}{\partial z_3})|_{c_2(t_2)} & (z_4 \frac{\partial f_0)}{\partial z_4})|_{c_2(t_2)} \end{vmatrix} \neq 0. \tag{4.28}$$

Therefore

$$Jac(f_0, c_2) \neq 0.$$

We complete the proof of lemma 4.2.

□

4.2 Ranks of differential sheaves

Proof. of proposition 1.4: Let $\mathcal{N}$ be the submodule of global sections, $H^0(\Omega_M)$ generated by elements

$$\phi_i = d \begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \\ f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \end{vmatrix} \tag{4.29}$$

for $i = 3, \cdots, 5d+1$. Recall that

$$\begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \\ f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \end{vmatrix} = 0,$$

for $i = 3, \cdots, 5d+1$ define the scheme $P(\Gamma_{\mathbb{L}})$ for a small $\mathbb{L}$. By proposition 8.12 in [7], II,

$$(\widetilde{\frac{H^0(\Omega_M)}{\mathcal{N}}}) \otimes \mathcal{O}_{P(\Gamma_{\mathbb{L}})} \simeq \Omega_{P(\Gamma_{\mathbb{L}})}, \tag{4.30}$$

where $\widetilde{(\cdot)}$ denotes the sheaf associated to the module $(\cdot)$.

Therefore

$$\begin{aligned} (\tfrac{H^0(\Omega_M)\otimes k(c_g)}{\mathcal{N}\otimes k(c_g)}) &\simeq \Omega_{P(\Gamma_{\mathbb{L}})} \otimes k(c_g) \\ &= (\Omega_{P(\Gamma_{\mathbb{L}})})|_{(\{c_g\})}, \end{aligned} \tag{4.31}$$

where $k(c_g) = \mathbb{C}$ is the residue field at generic

$$c_g \in P(\Gamma_{\mathbb{L}}).$$

Notice two sides of (4.31) are finitely dimensional linear spaces over $\mathbb{C}$.

$$\begin{aligned} & dim_{\mathbb{C}}((\Omega_{P(\Gamma_{\mathbb{L}})})|_{(\{c_g\})}) \\ &= dim_{\mathbb{C}}(H^0(\Omega_M) \otimes k(c_g)) - dim(\mathcal{N} \otimes k(c_g)) \end{aligned} \tag{4.32}$$

Since

$$(4.33) \qquad dim_{\mathbb{C}}((\Omega_{P(\Gamma_{\mathbb{L}})})|_{(\{c_g\})})) = dim(T_{c_g}P(\Gamma_{\mathbb{L}}))$$

$$(4.34) \qquad dim(T_{c_g}P(\Gamma_{\mathbb{L}})) = dim(M) - dim(\mathcal{N} \otimes k(c_g)).$$

By lemma 4.1,

$$dim(\mathcal{N} \otimes k(c_g)) = deg(\omega(M, \mathbf{t})).$$

The proposition 1.4 is proved.

□

Proof. of theorem 1.1. This is the statement of proposition 1.5. See section 3, [9] for the details. □

5 Examples –Vainsencher's and Chen's rational curves

Example 5.1 (Vainsencher's rational curves)

This example provides an evidence to theorem 1.1. In [8], Vainsencher constructed irreducible, degree 5, nodal curves C_0 on a generic quintic f_0 by taking plane sections of the quintic. Let c_0 be its normalization. By our theorem 1.1, c_0 is an immersion and

$$(5.1) \qquad N_{c_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

Indeed these were proved by Cox and Katz in [4], by using a different method. Their method is based on Clemens' deformation idea. Their understanding of c_0 on f_0 was achieved by a concrete construction of special c_0, f_0 and by using a computer program for the last verification of the 26×30 matrix. It is easy to check that the rational maps c_0 they constructed are immersions.

Furthermore our result shows

$$(5.2) \qquad dim(T_{c_0}\Gamma_{f_0}) = 4.$$

Because of the equation (5.1), C_0 can't deform in f_0. Thus Γ_{f_0} consists of multiple orbits isomorphic to $GL(2)(c_0)$. Theorem 1.1 also shows that there will not be any scheme-theoretical multiplicity associated to the orbits. However the number of these orbits is not accessible because the degree of each orbit in $\mathbf{P}(M)$ could be different. This number is related to Gromov-Witten invariants.

Example 5.2 (Chen's rational curves)

This is an example on K-3 surfaces. In [1], Chen constructed nodal rational curves C_0 of degree $4d$ for each natural number d, that lie on the generic hypersurfaces f_0 of degree 4 in $\mathbf{P}^3$ (f_0 is a K-3 surface). At first we may have an impression that this is against our intuition. Because it is similar to rational curves on generic quintic threefolds that we can have naive counting: on a generic quartic hypersurface f_0 of $\mathbf{P}^3$, there will be $4d+1$ conditions imposed the rational curves on f_0, while the dimension of the moduli space of rational curves in $\mathbf{P}^3$ (modulo $PGL(2)$ action) is

only $4d$. Thus the naive counting concludes that there will not be any rational curves on f_0. But it was proved by Mori, Mukai, etc., and Chen ([1]) that rational curves on f_0 exist and they are all nodal. Our proof is closely related to this counting, and our construction of $\omega(M, \mathbf{t})$ can be carried out in $\mathbf{P}^3$ for Chen's case. But theorem 1.1 does not hold because proposition 1.3 fails. This failure is not expected by the naive dimension count, but it is a reminder of a fact that the generic quartics are not generic in the moduli space of complex structures.

Chen's construction has a similar flavor of Vainsencher's rational curves above. They were obtained by taking hyperplane sections of K-3 surfaces. Intrinsically Vainsecher's and Chen's rational curves look similar. For instance they are all plane sections, and are all immersed, nodal rational curves. So what invariant distinguishes one from the other? Section 4 shows that this invariant may not be the invariant of the intrinsic rational curves, it addresses the structure of the moduli space of rational curves for underlined families of varieties. More specifically, it is deduced from the differential form $\omega(M, \mathbf{t})$ (defined in (1.10)). The ω itself is not a moduli invariant, but the zero locus $\{\omega(M, \mathbf{t}) = 0\}$ is, and furthermore $\{\omega(M, \mathbf{t}) = 0\}$ is independent of generic $t_i, i = 1, \cdots, 5d + 1$. In Chen's situation, $\omega(M, \mathbf{t})$ turns out to be identically zero on $P(\Gamma_{\mathbb{L}})$, but in Vainsencher's it is not. Beyond Chen's cases, it is not clear that which homology classes of rational curves would have or would not have vanishing $\omega(M, \mathbf{t})$.

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